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# Large-Deviations Estimates in Burgers Turbulence with Stable Noise Initial Data

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We consider the inviscid Burgers equation where the initial datum is given by a stable (Lévy) noise. The asymptotic behavior of the tail distribution of the solution is described; the decay is much faster in the case when the stable noise is completely skewed to the left.

**KEY WORDS:** Inviscid Burgers equation; random initial velocity; large deviations.

## 1. INTRODUCTION

The purpose of this paper is to solve a problem of large deviations involving Burgers turbulence for a certain class of random initial data. It has been motivated by the work of Avellaneda and E[1,2] who have obtained bonds for the tail distributions of the (Hopf–Cole) solution in the case when the initial velocity is given by a Gaussian white noise. We refer to Burgers,[4] Chorin,[5] and Woyczynski[19] and the references therein for physical motivations for such studies (and much more).

Before presenting our setting, let us recall some basic features in one-dimensional Burgers turbulence. Burgers equation with viscosity parameter $\varepsilon > 0$

$$\partial_t u + \partial_x (u^2/2) = \varepsilon\, \partial^2_{xx} u \tag{1}$$

has been solved explicitly by Hopf[11] and Cole[6] in terms of the initial condition $u(x, t=0)$. Loosely speaking, the solution $u_\varepsilon$ to (1) converges as the viscosity parameter $\varepsilon$ goes to 0 to some function $u_0 = u$, which is known

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as the Hopf–Cole solution to the inviscid Burgers equation. There are at least two useful expressions for this solution in terms of the initial datum $u(\cdot, 0)$. First, if we introduce a potential function $\psi$ by $u = -\partial_x \psi$, then for $t > 0$

$$\psi(x, t) = \sup_{a \in \mathbb{R}} \left\{ \psi(a, 0) - \frac{(x-a)^2}{2t} \right\} \tag{2}$$

which yields $u(x, t)$ by taking the $x$-derivative. Of course, (2) makes sense whenever the initial potential is such that $\psi(a, 0) = o(a^2)$ as $|a| \to \infty$, a condition that we implicitly take for granted. Second, if we denote by $a(x, t)$ the largest location $a$ for which the supremum in (2) is achieved, then the solution $u$ is given by

$$u(x, t) = \frac{x - a(x, t)}{t} \tag{3}$$

The function $a(x, t)$ is called the inverse Lagrangian function and its inverse (when the time $t$ is fixed and the location $x$ varies) the Lagrangian function. The Lagrangian function $x(a, t)$ gives the location at time $t$ of the fluid particle started initially at $a$ when the inviscid Burgers equation is used as a simplified model for hydrodynamic turbulence; see ref. 18. In this framework, $u(x, t)$ represents the velocity of the fluid particle located at $x$ at time $t$.

Burgers[4] has considered the situation when the initial datum is a white noise process, i.e., when the initial potential $\psi(\cdot, 0)$ is given by a (two-sided) Brownian motion. We refer again to Woyczynski[19] and the references therein for literature on the subject. The starting point of this work is an estimate of the tail distribution of the velocity obtained by Avellaneda and E (cf. Theorem 3 in ref. 1 and Theorem 1 in ref. 2):

$$C \exp\{-\tfrac{9}{2} t y^3\} \leqslant \mathbb{P}(u(x, t) > y) \leqslant C' \exp\{-\tfrac{1}{8} t y^3\}$$

It can be easily seen that $(u(x, t): x \in \mathbb{R})$ is a stationary process (cf. Section 2 in ref. 1), so the probability above does not depend on $x$. These bounds can be sharpened; we will show using a classical argument of large deviations that

$$-\log \mathbb{P}(u(x, t) > y) \sim \tfrac{1}{6} t y^3, \qquad y \to \infty \tag{4}$$

More precisely, (4) has been obtained first by Reade Ryan who has an even sharper estimate, see Theorem 1.1 in ref. 16.

The main purpose of this paper is to investigate the analogue of (4) when the initial velocity is a stable noise, i.e., when the initial potential is a (two-sided) stable Lévy process. In other words, $(\psi(x, 0), x \in \mathbb{R})$ has independent and stationary increments, and fulfils the scaling property

$$\psi(x, 0) \overset{law}{=} x^{1/\alpha} \psi(1, 0), \qquad x > 0$$

for some $\alpha \in (1/2, 2]$ known as the index. The restriction $\alpha > 1/2$ is due to the requirement that $\psi(a, 0) = o(a^2)$ as $|a| \to \infty$. The celebrated Lévy–Itô decomposition of non-Gaussian stable Lévy processes also enables us to describe the initial condition directly in terms of $u(\cdot, 0)$. Typically, for $\alpha < 2$, the initial velocity is a mixture of Dirac point masses, i.e., $u(x, 0) = w(x)\, \varepsilon_x$ where $\varepsilon_x$ denotes the Dirac point mass at $x$ and the weight $(w(x), x \in \mathbb{R})$ a certain Poisson point process. More precisely, the points $(x, w(x))$ form a Poisson cloud in $\mathbb{R}^2$ with intensity

$$(c_+ \mathbf{1}_{\{y > 0\}} + c_- \mathbf{1}_{\{y < 0\}})\, |y|^{-\alpha - 1}\, dx\, dy$$

for some nonnegative constants $c_+$ and $c_-$; see for instance Sections I.1 and VIII.1 in ref. 3.

The motivation for considering such type of initial condition stems from the key role of stable Lévy processes in a wide family of renormalized potentials. The best known example is that field is of course connected to sums of i.i.d. variables and domains of attraction; cf. Gnedenko and Kolmogorov.[10] For instance, suppose that the initial potential is given by $\psi(y, 0) - \psi(x, 0) = \sum_{x < j \leq y} \xi_j$ where $(\xi_j : j \in \mathbb{Z})$ is a family of i.i.d. variables in the normal domain of attraction of a stable law of index $\alpha$. This means that for each fixed $x \in \mathbb{R}$, $n^{-1/\alpha} \psi(xn, 0)$ converges in law as $n \to \infty$ to some $\alpha$-stable law. In this situation, a stronger result holds, namely the process $(n^{-1/\alpha} \psi(xn, 0), x \in \mathbb{R})$ converges weakly for Skorohod's topology towards a stable Lévy process, say $(\Psi(x, 0), x \in \mathbb{R})$; see for instance Jacod and Shiryaev.[13] It is then easily seen that

$$n^{-1/\alpha} \psi(nx, n^{2 - 1/\alpha} t) = \sup_{a \in \mathbb{R}} \left\{ n^{-1/\alpha} \psi(a, 0) - \frac{(nx - a)^2}{2tn^2} \right\}$$

converges in distribution as $n \to \infty$ towards

$$\sup_{a \in \mathbb{R}} \left( \Psi(x, 0) - \frac{(x - a)^2}{2t} \right)$$

which is the potential at time $t$ of the Hopf–Cole solution to the inviscid Burgers equation with initial potential $\Psi(\cdot, 0)$. Similar limit results can be obtained in the more general situation when the initial potential is expressed as the partial sum of some stationary sequence of variables in the domain of attraction of a stable law; see Ibragimov and Linnik[12] and also Davis and Resnick[7] in the particular case of moving averages. We also mention that there is a great variety of limit theorems in Burgers turbulence; see e.g., refs. 9, 15, and 19 for important results which do not involve stable noise. Finally, we refer to a recent work of Janicki and Woyczynski[14] where the inviscid Burgers equation with a stable Lévy initial velocity is considered, and to the book by Samorodnitsky and Taqqu[17] for a treatise on stable processes.

Our results depend crucially on the skew of the stable noise.

**Theorem 1.** Suppose that the initial potential $\psi(\cdot, 0)$ is a stable Lévy process with index $\alpha \in (1, 2]$, which is completely skewed to the left (i.e., there are no positive jumps). Its Laplace transform has the form

$$\mathbb{E}(\exp\{\lambda\psi(1, 0)\}) = \exp\{c\lambda^{\alpha}\}, \qquad \lambda > 0$$

for some $c > 0$. Then for every $t > 0$ and $x \in \mathbb{R}$, we have

$$-\log \mathbb{P}(u(x, t) > y) \sim c^{-1/(\alpha-1)}(\alpha-1)^2 (2\alpha-1)^{-1} \alpha^{-\alpha/(\alpha-1)} t y^{(2\alpha-1)/(\alpha-1)}, \qquad y \to \infty$$

The restriction $\alpha > 1$ on the index is due to the fact that in the case completely skewed to the left, the initial potential decreases when $\alpha < 1$, which implies that the velocity $u(x, t)$ is necessarily negative (and for $\alpha = 1$, completely skewed processes do not fulfil the scaling property). It is also interesting to observe that the decay is faster when $\alpha$ is smaller.

The asymptotic behaviour of the tail distribution of the velocity is much different when the stable noise is not completely skewed to the left.

**Theorem 2.** Suppose that the initial potential $\psi(\cdot, 0)$ is a stable Lévy process of index $\alpha \in (1/2, 2)$, which is not completely skewed to the left (i.e., there exist positive jumps). The characteristic function has the form

$$\mathbb{E}(\exp\{i\lambda\psi(1, 0)\}) = \exp\{-c\,|\lambda|^{\alpha}(1 - i\,\mathrm{sgn}(\lambda)\,\beta\tan(\pi\alpha/2))\}$$

for some $c > 0$ and $\beta \in (-1, 1]$ (for $\alpha = 1$ we agree that $\beta\tan(\pi\alpha/2) = 0$). Then for every $t > 0$ and $x \in \mathbb{R}$, we have

$$\kappa \leqslant \liminf_{y \to \infty} y^{2\alpha-1}\mathbb{P}(u(x, t) > y) \leqslant \limsup_{y \to \infty} y^{2\alpha-1}\mathbb{P}(u(x, t) > y) \leqslant 2\alpha\kappa$$

where

$$\kappa = \frac{c2^{\alpha-1}(1-\alpha)(1+\beta)}{(2\alpha-1)\,\Gamma(2-\alpha)\cos(\pi\alpha/2)}\,t^{\alpha-1} \qquad \text{for} \quad \alpha \neq 1$$

and $\kappa = c/\pi$ for $\alpha = 1$.

We thus see that the decay is now faster when $\alpha$ is larger, and in any case, much slower than when the noise is completely skewed to the left. It is interesting to point out that the mean of the positive part of the velocity is then infinite; in particular this impedes the application of the ergodic theorem to the potential $\psi(x, t) = -\int_0^x u(y, t)\,dy$.

Finally, we mention that the arguments of Avellaneda and E also apply in the present setting and enables us to deduce from Theorems 1 and 2 upper bounds for the tail distributions of the so-called rarefaction intervals and shock-strength; see Section 5 in ref. 1. Alternatively, estimates for the Lagrangian function follow easily since

$$\mathbb{P}(x(0, t) > y) = \mathbb{P}(a(y, t) < 0) = \mathbb{P}(u(y, t) > y/t) = \mathbb{P}(u(0, t) > y/t)$$

## 2. PROOFS

It is easily seen that $u(\cdot, t)$ is a stationary process, so we may take $x = 0$. On the other hand, it follows from the scaling property that for every $t > 0$

$$u(0, t) \overset{law}{=} t^{(1-\alpha)/(2\alpha-1)} u(0, 1)$$

(cf. Eq. (21) in ref. 18), so we only consider the case $t = 1$.

Of course, Theorem 1 is an extension of (4); the latter corresponds to the special case $\alpha = 2$ and $c = 1/2$, and follows from Theorem 1.1 in ref. 16. Nonetheless, it may be worthwhile to present its direct proof as it uses standard large deviations arguments that cannot be applied entirely in the general stable case, but provide valuable guidelines.

*Proof of (4).* We write $B_a = \psi(a, 0)$ for the initial potential, so $(B_a, a \geq 0)$ and $(B_{-a}, a \geq 0)$ are two independent standard Brownian motions.

We start with the upper bound. According to (3), the event $\{u(0, 1) > y\}$ means that the largest location of the maximum of $B_a - \frac{1}{2}a^2$ is greater than $y$. So introduce the stopping time

$$T_y = \inf\{a > y : B_a - \tfrac{1}{2}a^2 \geq \sup_{0 \leq b \leq y} (B_b - \tfrac{1}{2}b^2)\}$$

The process

$$\exp\left\{\int_0^a s\,dB_s - \tfrac{1}{6}a^3\right\}, \qquad a \geqslant 0$$

is a positive martingale. By an integration by parts, it can also be written as

$$\exp\left\{aB_a - \int_0^a B_s\,ds - \tfrac{1}{6}a^3\right\}, \qquad a \geqslant 0$$

On the event $\{u(0, 1) > y\} = \{T_y < \infty\}$, we put $A = B_{T_y} - \frac{1}{2}T_y^2$. In particular $B_a - \frac{1}{2}a^2 \leqslant A$ for all $a \leqslant T_y$ and hence $\int_0^{T_y} B_s\,ds \leqslant AT_y + \frac{1}{6}T_y^3$. An application of the optional sampling theorem yields

$$\begin{aligned} 1 &\geqslant \mathbb{E}(\exp\{T_y B_{T_y} - AT_y - \tfrac{1}{6}T_y^3 - \tfrac{1}{6}T_y^3\}, T_y < \infty) \\ &= \mathbb{E}(\exp\{\tfrac{1}{6}T_y^3\}, T_y < \infty) \\ &\geqslant \exp\{\tfrac{1}{6}y^3\}\ \mathbb{P}(T_y < \infty) \end{aligned}$$

which proves the upper bound.

To establish the lower bound, we will use Girsanov's Theorem and to that end, we introduce the probability measure $\mathbb{Q}$ given by

$$d\mathbb{Q}\,|_{\mathscr{F}_a} = \exp\left\{aB_a - \int_0^a B_s\,ds - \tfrac{1}{6}a^3\right\} d\mathbb{P}\,|_{\mathscr{F}_a}, \qquad a \geqslant 0$$

where $\mathscr{F}_a$ denotes the natural filtration of $B$. We consider the events

$$\Lambda_y = \{|B_a - \tfrac{1}{2}a^2| \leqslant 1 \text{ for all } a \leqslant y\} \qquad \text{and} \qquad \Lambda'_y = \{B_{y+1} - B_y \geqslant y + 3\}$$

which are clearly independent. Note that on $\Lambda_y \cap \Lambda'_y$, we have both $\sup_{a \geqslant 0}(B_a - \frac{1}{2}a^2) \geqslant 1$ and the location of this supremum is greater than $y$. Let $p > 0$ denote the probability that $\sup_{a \leqslant 0}(B_a - \frac{1}{2}a^2) \leqslant 1$. We thus have

$$\mathbb{P}(u(0, 1) > y) \geqslant p\,\mathbb{P}(\Lambda_y)\,\mathbb{P}(\Lambda'_y)$$

On the one hand, it is plain that $-\log \mathbb{P}(\Lambda'_y) \asymp y^2$ as $y \to \infty$. On the other hand,

$$\mathbb{P}(\Lambda_y) = \mathbb{Q}\left(\exp\left\{-yB_y + \int_0^y B_s\,ds + \tfrac{1}{6}y^3\right\}, \Lambda_y\right)$$

Note that on $\Lambda_y$,

$$-yB_y + \int_0^y B_s\, ds + \tfrac{1}{6}y^3 \geqslant -\tfrac{1}{6}y^3 - 2y$$

so the preceding probability is bounded from below by $\exp\{-\frac{1}{6}y^3 - 2y\} \times \mathbb{Q}(\Lambda_y)$. But we know from Girsanov's Theorem that $B_a - \frac{1}{2}a^2$ is a $\mathbb{Q}$-Brownian motion, and thus $-\log \mathbb{Q}(\Lambda_y) \asymp y$. Putting the pieces together completes the proof of the lower bound. ∎

We next turn our attention to the stable case. It will be convenient to write $X_a = \psi(a, 0)$ for the initial potential, i.e., $(X_a, a \geqslant 0)$ and $(-X_{-a}, a \geqslant 0)$ are two independent identically distributed stable Lévy processes. We first consider the case when the stable noise is completely skewed to the left.

*Proof of the Upper Bound in Theorem 1.* The proof is very similar to that in the Gaussian case $\alpha = 2$. Specifically, let $\gamma$ and $\rho$ be two positive real numbers that will be chosen later on. We first consider the exponential martingale

$$\exp\left\{\gamma \int_0^a s^\rho\, dX_s - \frac{c\gamma^\alpha a^{\alpha\rho+1}}{\alpha\rho+1}\right\}, \qquad a \geqslant 0$$

By an integration by parts (cf. for instance Theorem VIII.19 on p. 343 in ref. 8), we can rewrite this as

$$\exp\left\{\gamma a^\rho X_a - \gamma\rho \int_0^a X_s s^{\rho-1}\, ds - \frac{c\gamma^\alpha a^{\alpha\rho+1}}{\alpha\rho+1}\right\}, \qquad a \geqslant 0$$

We then introduce the stopping time

$$T_y = \inf\{a \geqslant y : X_a - \tfrac{1}{2}a^2 \geqslant \sup_{0 \leqslant b \leqslant y} (X_b - \tfrac{1}{2}b^2)\}$$

and put $A = X_{T_y} - \frac{1}{2}T_y^2$ on the event $\{u(0, 1) > y\} = \{T_y < \infty\}$. The inequality $X_a \leqslant A + \frac{1}{2}a^2$ for $0 \leqslant a \leqslant T_y$ entails

$$\gamma\rho \int_0^{T_y} X_s s^{\rho-1}\, ds \leqslant \gamma\left(A T_y^\rho + \frac{\rho}{2(\rho+2)} T^{\rho+2}\right)$$

Just as in the proof of (4), we then deduce from an application of the optional sampling theorem that

$$\mathbb{E}\left(\exp\left\{\frac{\gamma}{2}T_y^{\rho+2}-\frac{\gamma\rho}{2(\rho+2)}T_y^{\rho+2}-\frac{c\gamma^\alpha}{\alpha\rho+1}T_y^{\alpha\rho+1}\right\}, T_y<\infty\right)\leqslant 1$$

We first tune up $\rho$ by considering the exponents of $T_y$. We see that the optimal inequality is obtained when $\rho+2=\alpha\rho+1$, that is $\rho=1/(\alpha-1)$. As $T_y\geqslant y$, this yields

$$\mathbb{P}(T_y<\infty)\leqslant\exp\{-(\gamma-c\gamma^\alpha)(\alpha-1)(2\alpha-1)^{-1}y^{(2\alpha-1)/(\alpha-1)}\}$$

We then tune up $\gamma$, the best inequality is obtained when $\gamma=(c\alpha)^{-1/(\alpha-1)}$. So $\gamma-c\gamma^\alpha=c^{-1/(\alpha-1)}(\alpha-1)\,\alpha^{-\alpha/(\alpha-1)}$, which proves the upper bound in Theorem 1. ∎

*Proof of the Lower Bound in Theorem 1.* In the stable case, we can no longer rely on Girsanov's Theorem as in the Brownian case. However the idea is similar, in the sense that we shall focus on the set of paths of the stable process whose supremum at time $a$ is close to $\frac{1}{2}a^2$ for $0\leqslant a\leqslant y$, and evaluate the probability of this event by bare hands methods.

More precisely, introduce the first passage process

$$\tau_a=\inf\{b\geqslant 0: X_b>a\},\qquad a\geqslant 0$$

It is known that $\tau=(\tau_a, a\geqslant 0)$ is a stable subordinator (i.e., increasing Lévy process) with index $1/\alpha$; more precisely

$$\mathbb{E}(\exp\{-\lambda\tau_a\})=\exp\{-a(\lambda/c)^{1/\alpha}\},\qquad a,\lambda\geqslant 0$$

See e.g., Theorem VII.1 in ref. 3. Recall also that the distribution of $\tau_1$ is absolutely continuous with a continuous density $p(x)$ which has

$$p(x)\geqslant\exp\{-c^{-1/(\alpha-1)}(\alpha-1)\,\alpha^{-\alpha/(\alpha-1)}x^{-1/(\alpha-1)}\}\tag{5}$$

for small enough $x>0$. See Eq. (2.5.18) in Zolotarev.[20]

We then consider the sequence of independent events

$$\Lambda_n=\left\{\frac{1}{\sqrt{2(n+1)}}\leqslant\tau_n-\tau_{n-1}\leqslant\frac{1}{\sqrt{2n}}\right\},\qquad n\geqslant 1$$

An application of (5) shows that for large enough $n$

$$\mathbb{P}(\Lambda_n)\geqslant n^{-2}\exp\{-c^{-1/(\alpha-1)}(\alpha-1)\,\alpha^{-\alpha/(\alpha-1)}(2(n+1))^{1/(2\alpha-2)}\}$$

After some elementary calculations, one deduces that for every fixed $\varepsilon > 0$, we have

$$\log \mathbb{P}(A_1 \cap \cdots \cap A_{(1/2)n^2}) \geqslant -(1+\varepsilon)\, c^{-1/(\alpha-1)}(\alpha-1)^2\,(2\alpha-1)^{-1}\,\alpha^{-\alpha/(\alpha-1)} n^{(2\alpha-1)/(\alpha-1)} \tag{6}$$

provided that $n$ being a large enough even integer.

The inequality

$$\frac{1}{\sqrt{2(k+1)}} \leqslant \sqrt{2(k+1)} - \sqrt{2k} \leqslant \frac{1}{\sqrt{2k}}$$

entails that on $A_1 \cap \cdots \cap A_{(1/2)n^2}$, we have

$$\sqrt{2(k+2)} - 2 \leqslant \tau_k \leqslant \sqrt{2k}, \qquad k = 1,..., \tfrac{1}{2}n^2 \tag{7}$$

The lower bound means that $X_a \leqslant k$ for $0 \leqslant a \leqslant \sqrt{2(k+2)} - 2$ and $k = 1,..., \frac{1}{2}n^2$. In particular, observing that for $\sqrt{2(k+1)} - 2 \leqslant a \leqslant \sqrt{2(k+2)} - 2$ one has $k \leqslant \frac{1}{2}(a+2)^2$, this yields

$$X_a \leqslant \tfrac{1}{2}a^2 + 2a + 2 \qquad \text{for} \quad 0 \leqslant a \leqslant \sqrt{n^2+4} - 2$$

and *a fortiori*

$$X_a - \tfrac{1}{2}a^2 \leqslant 2n - 2 \qquad \text{for} \quad 0 \leqslant a \leqslant n - 2$$

On the other hand, recall from (7) that $\tau_{(1/2)n^2} \leqslant n$ on $A_1 \cap \cdots \cap A_{(1/2)n^2}$, so that

$$\begin{aligned}
&\mathbb{P}(\exists a \leqslant n+1 : X_a - \tfrac{1}{2}a^2 > 2n \mid A_1 \cap \cdots \cap A_{(1/2)n^2}) \\
&\quad \geqslant \mathbb{P}(\exists a \leqslant n+1 : X_a > \tfrac{1}{2}n^2 + 3n + \tfrac{1}{2} \mid A_1 \cap \cdots \cap A_{(1/2)n^2}) \\
&\quad = \mathbb{P}(\tau_{(1/2)n^2+3n+1/2} \leqslant n+1 \mid A_1 \cap \cdots \cap A_{(1/2)n^2}) \\
&\quad \geqslant \mathbb{P}(\tau_{(1/2)n^2+3n+1/2} - \tau_{(1/2)n^2} \leqslant 1 \mid A_1 \cap \cdots \cap A_{(1/2)n^2}) \\
&\quad = \mathbb{P}(\tau_{3n+1/2} \leqslant 1)
\end{aligned}$$

By the scaling property, the ultimate quantity equals the probability that $\tau_1 \leqslant (3n+\frac{1}{2})^{-\alpha}$; and it follows from (5) that this is larger that $\exp\{-kn^{\alpha/(\alpha-1)}\}$ for some $k > 0$.

Plainly, the event $\{\sup_{a \leqslant 0}(X_a - \frac{1}{2}a^2) < 2n-2\}$ is independent of $(X_a, a \geqslant 0)$ and its probability tends to 1 as $n \to \infty$. We conclude from (6) that for $n$ large enough, the logarithm of the probability that $X_a - \frac{1}{2}a^2 < 2n-2$ for $-\infty < a \leqslant n-2$ and $X_a - \frac{1}{2}a^2 > 2n$ for some $a \leqslant n+1$ is at least

$$-(1+2\varepsilon)\, c^{-1/(\alpha-1)}(\alpha-1)^2\,(2\alpha-1)^{-1}\,\alpha^{-\alpha/(\alpha-1)} n^{(2\alpha-1)/(\alpha-1)}$$

(Recall that $\alpha > 1$, so $\alpha/(\alpha-1) < (2\alpha-1)/(\alpha-1)$.) On the aforementioned event, we have clearly $u(0,1) \geqslant n-2$, so

$$\liminf_{n \to \infty} n^{-(2\alpha-1)/(\alpha-1)} \log \mathbb{P}(u(0,1) > n-2)$$
$$\geqslant -(1+2\varepsilon)\, c^{-1/(\alpha-1)}(\alpha-1)^2\,(2\alpha-1)^{-1}\,\alpha^{-\alpha/(\alpha-1)}$$

As $\varepsilon > 0$ is arbitrary, this establishes the lower bound in Theorem 1. ∎

We next consider the case non completely skewed to the left. We shall only consider the case $\alpha \neq 1$ as the Cauchy case $\alpha = 1$ only requires obvious notational changes.

*Proof of the Upper Bound in Theorem 2.* We first observe that

$$\mathbb{P}(u(0,1) > y) \leqslant \mathbb{P}(\sup_{a \geqslant y}(X_a - \tfrac{1}{2}a^2) \geqslant 0) = \mathbb{P}(\sup_{a \geqslant y}(a^{-2}X_a) \geqslant \tfrac{1}{2})$$
$$= \mathbb{P}(\sup_{a \geqslant 1}(a^{-2}X_a) \geqslant \tfrac{1}{2} y^{2-1/\alpha})$$

where the ultimate identity follows from the scaling property. Now the process $(a^{-2}X_a, a \geqslant 1)$ is a stable (non-Lévy) process which can be expressed in the form

$$a^{-2}X_a = \int_{[1,\infty)} f(a,x)\, M(dx)$$

where $M$ is an $\alpha$-stable random measure on $[1,\infty)$ with control measure $m(dx) = c(\varepsilon_1(dx) + dx)$ and constant skewness intensity $\beta$, and $f(a,x) = a^{-2}\mathbf{1}_{\{1 \leqslant x \leqslant a\}}$. See Section 3.3 in ref. 17. An application of Theorem 10.5.1 in ref. 17 yields

$$\mathbb{P}\left(\sup_{a \geqslant 1}(a^{-2}X_a) \geqslant \frac{1}{2} y^{2-1/\alpha}\right) \sim \frac{\alpha c 2^{\alpha}(1-\alpha)(1+\beta)}{(2\alpha-1)\,\Gamma(2-\alpha)\cos(\pi\alpha/2)}\, y^{1-2\alpha}$$

which entails the desired upper bound. ∎

*Proof of the Lower Bound in Theorem 2.* Fix $\varepsilon > 0$ and consider for each integer $n \geqslant 1$ the events

$$A'_n = \{ \sup_{0 \leqslant a \leqslant n} |X_a| \leqslant \varepsilon n^2 \}$$

$$A''_n = \{ \sup_{0 \leqslant a \leqslant 1} (X_{n+a} - X_n) \geqslant (\tfrac{1}{2} + 2\varepsilon)(n+1)^2 \}$$

Note that $A'_n$ and $A''_n$ are independent for each fixed $n$, and that $A''_1,..., A''_n,...$ are also independent. We then introduce

$$A_n = A'_n \cap A''_n$$

Pick an arbitrarily large integer $n_0$ and observe that for each $n \geqslant n_0$, on the event $A_n$, the location of the supremum for $a \geqslant 0$ of $X_a - \frac{1}{2}a^2$ is larger than $n \geqslant n_0$, and the value of this supremum is at least $\varepsilon n^2 \geqslant \varepsilon n_0^2$. Plainly the event $\{\sup_{a \leqslant 0}(X_a - \frac{1}{2}a^2) < \varepsilon n_0^2\}$ is independent of the $A_n$'s and has a probability which goes to 1 as $n_0 \to \infty$. We deduce that

$$\liminf_{n_0 \to \infty} n_0^{2\alpha - 1} \mathbb{P}(u(0, 1) \geqslant n_0) \geqslant \liminf_{n_0 \to \infty} n_0^{2\alpha - 1} \mathbb{P}\Big( \bigcup_{n \geqslant n_0} A_n \Big)$$

In order to estimate the quantity in the right-end side, we use the classical inequality

$$\mathbb{P}\Big( \bigcup_{n \geqslant n_0} A_n \Big) \geqslant \frac{(\sum_{n=n_0}^{\infty} \mathbb{P}(A_n))^2}{\sum_{m=n_0}^{\infty} \sum_{n=n_0}^{\infty} \mathbb{P}(A_n \cap A_m)}$$

which follows from an application of Cauchy-Schwarz inequality to $\sum \mathbf{1}_{A_n} = \mathbf{1}_A(\sum \mathbf{1}_{A_n})$ with $A = \bigcup A_n$. On the one hand, it is plain from the scaling property that $\mathbb{P}(A'_n)$ tends to 1 as $n \to \infty$, and it is known that

$$\mathbb{P}(A''_n) \sim \frac{c(1-\alpha)(1+\beta)}{2\Gamma(2-\alpha)\cos(\pi\alpha/2)} \left(\frac{1}{2} + 2\varepsilon\right)^{-\alpha} n^{-2\alpha}, \qquad n \to \infty \tag{8}$$

see e.g., again Theorem 10.5.1 in ref. 17. It follows that as $n_0 \to \infty$,

$$\sum_{n=n_0}^{\infty} \mathbb{P}(A_n) \sim \sum_{n=n_0}^{\infty} \mathbb{P}(A''_n) \sim \frac{c(1-\alpha)(1+\beta)}{2(2\alpha-1)\,\Gamma(2-\alpha)\cos(\pi\alpha/2)} \left(\frac{1}{2} + 2\varepsilon\right)^{-\alpha} n_0^{1-2\alpha}$$

On the other hand, we have

$$\mathbb{P}(A_n \cap A_m) \leqslant \mathbb{P}(A''_n \cap A''_m) = \mathbb{P}(A''_n)\,\mathbb{P}(A''_m) \qquad \text{if} \quad n \neq m$$

We then use again (8) to deduce that

$$\sum_{m=n_0}^{\infty} \sum_{n=n_0}^{\infty} \mathbb{P}(\Lambda_n \cap \Lambda_m) \sim \sum_{n=n_0}^{\infty} \mathbb{P}(\Lambda_n), \qquad n_0 \to \infty$$

We now see by putting the pieces together that

$$\liminf_{n_0 \to \infty} n_0^{2\alpha-1} \mathbb{P}(u(0,1) \geqslant n_0) \geqslant \frac{c(1-\alpha)(1+\beta)}{2(2\alpha-1)\,\Gamma(2-\alpha)\cos(\pi\alpha/2)} \left(\frac{1}{2} + 2\varepsilon\right)^{-\alpha}$$

which in turn yields the desired lower bound as $\varepsilon > 0$ is arbitrary. ▮

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